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EQUAL TEMPERAMENT WITH THIRTY-ONE NOTES

IN the Middle Ages the relations of the musical notes were defined following the rule of Pythagoras. The series of notes fa — ut — sol — re — la — mi — si or, as we nowadays put them, F — C — G — D — A — E — B were conceived to make among themselves perfect fifths or perfect fourths, related to harmonic numbers 2 and 3, or 4 and 3. In this temperament the relation of the major third, between F and A, or C and E, G and B, was not very simple. Accordingly the major third was considered to be a discord.

In the century before the Renaissance, when polyphonic music from the Netherlands spread to the south, the perfect third, corresponding to harmonic numbers 4 and 5, was used as a valuable and beautiful concord. All went well in singing, but on the instruments with fixed pitches there was a clash. The great masters laying down the foundation for musical structure, Zarlino in Italy and Salinas in Spain, then ruled that in order to keep the major third perfect, the four fifths, leading from F to A, or from C to E, or from G to B, should be cut down a trifle. The twelve notes, available on the the keyboards of organs and harpsichords now can be arranged in a two-dimensional diagram, where the rows are transposed by a perfect third.

E♭ — B♭ — F — C — G — D — A — E — B — F♯ — C♯ — G♯
E♭ — B♭ — F — C — G — D — A — E — B — F♯ — C♯ — G♯
E♭ — B♭ — F — C — G — D — A — E — B — F♯ — C♯ — G♯

This temperament offered a choice of six major and five minor keys. There were seven major semitones in it, and five minor semitones. The variety of successions of major and minor semitones was a source of great beauty in the melodic lines. Above all, the common chords with perfect thirds offered delightful harmonies.

In course of time, however, the lack of full freedom of modulation was considered to be a serious shortcoming. If once the instrument had strings or pipes tuned to G♯, you could not produce a harmony requiring an A♭. Neither could E♭ be used for D♯, or any sharp for a nearby flat. There was a growing tendency to sacrifice the refinement of the variety of semitones in order to make them all equal and thereby to gain complete freedom of modulation.

The exact numbers dividing the octave into twelve equal steps, have been calculated in four significant decimal numbers as early as the beginning of the seventeenth century by Simon Stevin, of Brugge, Flanders. Stevin was a first-rate mathematician. He put into practice the decimal system, and he invented a method of extracting the cube root. That was why he could so well compute equal temperament in twelve steps.

From the middle of the nineteenth century all keyboard instruments gradually were tuned to the duodecimal equal temperament. This equal temperament has been rendering great service since then. Still, it might be useful to look for a better solution of the problem, set by the demand of full liberty of modulation, without sacrificing the beauties of the perfect thirds in the chords and the expressiveness of major and minor semitones.

Let us put in a cross two sets of seven notes, centering in D, as follows:

			C𝄪			
			A♯			
			F♯			
F	C	G	D	A	E	B
			B♭			
			G♭			
			E♭♭			

The horizontal row contains the Pythagorean notes a fifth apart. The vertical column contains a series of perfect thirds. In order to put fifths and thirds on equal footing, we take equal numbers of them in row and column. By adjusting the fifths according to Zarlino's rule we ensure that the fifth leading from B to F♯ produces the note one perfect third above D. Similarly the B♭ one third below D is one fifth below F.

One can now fill the table with new notes, proceeding from the ones already written down, by fifths to the right and to the left. The following is the result:

			C𝄪	G𝄪	D𝄪	A𝄪
			A♯	E♯	B♯	F𝄪
			F♯	C♯	G♯	D♯
F	C	G	D	A	E	B
D♭	A♭	E♭	B♭			
B♭♭	F♭	C♭	G♭			
G♭♭	D♭♭	A♭♭	E♭♭			

Thus one finds 31 notes in all. It was for Christiaan Huygens, mathematician and physicist and a fine musician, to discover that if one tries to extend the table any further, one gets back to notes already put down. One fifth from A𝄪, one third from C𝄪, there comes E𝄪. The first discovery of Huygens was that E𝄪 is very much the same note as G♭♭.* Likewise D𝄪 and F♭♭ are synonymous, A𝄪 and C♭♭, B𝄪 and D♭♭.

By the duodecimal temperament at present one is wont to put F for G♭♭ and F♯ for E𝄪. Huygens shows more refinement in this. It is the refinement which distinguishes between major and minor semitones. In his temperament there are 31 different notes, and no more. There is a series of 31 thirds, likewise leading from G♭♭ to E𝄪 (=G♭♭). Obviously the octave has to be divided in 31 equal steps. The octave having five whole tones and two major semitones, it follows that one whole tone, the second, contains five elementary steps. The major and the minor semitones will have three and two elementary steps, respectively. The octave will have three perfect thirds of ten steps each and one more elementary step. The elementary steps are fifths of a tone. Huygens called them *dieses.*

Compelled by the facts of the autochthonous music of his native countrymen, Aloys Haba, in Prague, felt the necessity of more subtle varieties of pitch. He tried to supply this need by introducing quarter tones. It must be emphasized, however, that this cannot lead to an improvement. It would be adding other false notes to a false temperament. The improvement is achieved by introducing fifths of a tone. This unit, 1/31 of an octave, measures 1200/31 = 38.7 cents.

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* From the table one verifies that 3 fifths and 7 thirds lead from G♭♭ to E𝄪. A fifth being 702 cents and a third 386 cents, the distance is 2,106 + 2,702=4,808 cents. This is only 8 cents more than 4 octaves. Quod erat demonstrandum.

The second discovery of Christiaan Huygens was that in this "tricesimoprimal" temperament the interval of the perfect seventh, related to harmonic numbers 4 and 7, is very accurately reproducible. It is found between pairs of notes, such as C and A♯. It takes 25 elementary steps in this temperament, that is 967.5 cents. The exact true value is 969 cents.

No attention was paid to this discovery until the edition of volume XX of Huygens' Oeuvres Complètes in 1942. It means that in the tricesimoprimal temperament the common chord with harmonic numbers 4, 5, 6, 8 can be completed to a primary tetrad chord with harmonic numbers 4, 5, 6, 7, 8. That is just the thing Béla Bartók wanted, when he stated that the study of the peasants' music of his native country, Hungary, led him to add the seventh to the common chord, the primary tetrad thus formed providing a foundation for his music. One now sees that Haba's need for more refinement points to a certain similarity in the music of the Czechish and the Hungarian people, in the use made of the harmony of the perfect seventh.

Turning from theory to the practice of instruments, the problem is to find a keyboard enabling one to play 31 different notes in the octave, all the 31 intervals inside the octave lying easily within the compass of the human hand. The 31-keyed organ now in Teyler's Museum at Haarlem, Netherlands, has two manuals with the keys arranged as follows.

In horizontal rows neighbouring keys lie one tone apart: C — D — E — F♯ — G♯ — A♯ — B♯ — C^{x} — D^{x} — E^{x} (= G♭♭) — A♭♭ — B♭♭ — C♭′ — D♭′ — E♭′ — F′ — G′ — A′ — B′ — C♯″ — D♯″ — E♯″ — F^{x}″. In sloping, so to speak "vertical" tiers, the keys lie one-fifth of a tone apart. There is a tier starting from C upward: C — D♭♭ — C♯ — D♭— C^{x} — D. The next one starts from D. It contains the keys for D — E♭♭ — D♯ — E♭ — D^{x} — E. Next comes the tier containing E — F♭ — E♯ — F — G♭♭ — F♯. And this continues.

Halfway between E and F♯ in the horizontal row, and ascending the slope halfway between E and F♭, one finds the key for F, halfway between F♭ and F♯. The horizontal row through F shows E♭ — D♭ to the left, and G — A — B — C♯ — D♯ — E♯ — F^{x}, and so on to the right.

Halfway between B and C♯, in the tier starting from B♯, one step above B♯, one finds another C.

From F again there is a rising tier: F — G♭♭ — F♯ — G♭ — F^{x}. This tier contains only five keys. Similar tiers slope upward from E♭ and D♭, from G, A, and so on.

There are skew tiers coming down to the right. The steps in these tiers are rising minor semitones: D — D♯ — D^{x} (=F♭♭) — F♭ — F♮ — F ♯ — F^{x} — A♭♭ — A♭ — A — A♯. Nine minor semitones make a fifth D — A. Other skew tiers are ascending to the right. They contain major semitones. There is one showing D — E♭ — F♭ — G♭♭ (= E^{x}) — F^{x} — G♯ — A — B♭ — C♭ — D♭♭ (= B^{x}) — C^{x}. Six major semitones make a fifth D — A.

The keys have been colored. In the octave there are seven white keys, for the basic notes A — B — C — D — E — F — G. For each black key in the ordinary keyboards there are two black keys here, separate ones for the flats and the sharps. C♯ and D♭ are two different black keys. On both sides up and down from a white key, there are blue keys. The white key for C is in between the blue keys for B♯ and for D♭♭. The white key for D is in between the blue for C^{x} and E♭♭. Similarly for the other white keys. Thus there are seven white, ten black, and fourteen blue keys for the 31 notes in the octave.

On the keyboard every note is represented by two, some notes by three keys, sounding the same pipe. Thus there is a choice for the most convenient fingering. One might say roughly there is one set of keys for the long fingers of the hand, and another set for use by the short fingers.

There is complete similitude of situation of the keys for any chord or any gamut whatever the pitch. Transposition is effected by simple displacement of the hand without change in the fingering. These two features: alternative choice of keys and perfect homogeneity of situation for chords in all pitches, contribute to facility in playing the organ.

In the equal temperament of 31, playing G — B — D′ — E♯′, the last note is the perfect seventh to G. For musicians, however, the notation E♯ has many associations, but it does not suggest the seventh to G. The seventh to G has to do with F, not with E. Therefore another notation would be useful. One finds it in the Trattato di Musica of Giuseppe Tartini (1754). Tartini introduced a sign for half a flat: a kind of hook. We accordingly can call the tetrad chord G — B — D — F♭ (half-flat). Instead of half-flat one can say F-minus. Again, Tartini offered a sign for one-and-a-half flat: a kind of contraction of the hook of one half-flat with a flat. Thus the chord with perfect seventh on C can be written, not with A♯, but with B-one-and-a-half flat or B-flat-minus, viz. C — E — G — B♭𝄳.

What has been done by Tartini in the flats can also be done in the sharps. One can introduce a half-sharp. In dealing with subharmonic tetrads, that might be useful. Such a chord is C — A♭ — F — D‡. This is the same as C — A♭ — F — E♭♭, but the suggestion of a seventh is more vivid with D‡ than with E♭♭. One may call that note D-plus. Likewise one can introduce one-and-a-half sharp, ♯‡. The subharmonic tetrad E — C — A — F♯‡ offers an instance with F-sharp-plus.

To conclude one may say that the tricesimoprimal equal temperament has two faces. One face is turned to the past. It recalls into being the all-but-forgotten beauties of the music of the great old masters by restoring the temperament they breathed in. The other face is turned to the future, with its possibility of coping with new harmonies and with autochthonous music outside the compass of traditional secular western music.

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