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THE HUYGENS COMMA: SOME MATHEMATICS CONCERNING THE 31-CYCLE

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1. Chains of pure fifths

As is well known, a chain of consecutive pure fifths (combined with an appropriate number of octave jumps) never brings us back to the starting tone. Mathematically this is expressed by the following:

$$\left(\frac{3}{2}\right)^m \neq 2^n \qquad (1)$$

that states that it is impossible to find two integers (*m; n*) such to satisfy the equality criterion (except the obvious trivial solution: $m = n = 0$).

However, pairs of integer numbers (*m; n*) can be found such that the following C_m approaches 1:

$$C_m = \frac{(3/2)^m}{2^n} \approx 1 \qquad (2)$$

The first sensible solution of Eq.(2) is found to be

$$m = 12\ ;\ n = 7 \ \Rightarrow\ C_{12} = 1.013643265\ \square\ C_P \qquad (3)$$

the well known *Pythagorean comma*. For example, this means that if we start from a note (say F) and go 12 fifths up and 7 octaves down, we find a note (E#) sharper than the starting one by a factor C_P. Still a perceptible difference.

The next more precise solutions of Eq.(2) are

$$m = 41\ ;\ n = 24 \ \Rightarrow\ C_{41} = 0.988602548 = (1.011528852)^{-1}$$

$$m = 53\ ;\ n = 31 \ \Rightarrow\ C_{53} = 1.002090314 \qquad (4)$$

C_{53} is also known as Mercator's comma, the last interval listed in the comprehensive compilation given in [1] (a difference that would yield only slow beats with the starting tone). We give the subsequent solution only as a mathematical curiosity:

$$m = 306\ ;\ n = 179 \ \Rightarrow\ C_{306} = 0.998978282 = (1.001022762)^{-1} \qquad (5)$$

Measuring the intervals in *cents*[1] one has[2]:

$$I(C_P) = 23.5c$$

$$I(C_{41}) = -19.8c \qquad (6)$$

$$I(C_{53}) = 3.6c$$

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[1] The measure in *cents* of an interval between two tones, whose frequency ratio is *R*, is defined as $I(R)=1200\cdot\log R/\log 2$. In this paper it will be sufficient to specify such measures up to one decimal place.

[2] Note that, since C_{41} is less than 1, a negative number corresponds to its interval measure $I(C_{41})$.

Thus, dividing the octave in 12, 41, or 53 equal parts amounts to achieving the first approximations to Pythagorean intervals and scales. In fact, the tuning in such a *12-tone* equal temperament[3] corresponds to the nowadays widespread equal temperament and is performed by narrowing each fifth by about 2*c* , in order to reach the exact starting tone over a cycle of 12 perfect fifths.

2. The 53-division

Let us proceed further to the division in 53 equal parts. In this case fifths are to be narrowed by an extremely tiny amount ($I(C_{53})/53 = 0.06c$) to close the cycle after 53 steps and one gets perfect Pythagorean scales in the sense that the intervals between notes cannot be distinguished from those obtained by chains of pure fifths. This division is known as *Mercator's division* and is represented in Figure 1. Of course, as we all know, Pythagorean major thirds are very wide, larger than pure thirds (ratio=5/4) by a syntonic (or Didymus or Zarlino) comma ($I(C_Z) = 21.5c$). As a consequence pure minor thirds (ratio=6/5) are also too narrow by the same amount. Since major thirds are obtained after a sequence of four fifths, they will be even wider in the 41-division (the comma C_{41} <1 implies a widening of the fifth); surely they sound a little better to our ears in the 12-division (though still not quite satisfying), because in that case major thirds are reduced by 7.8*c* with respect to the Pythagorean ones and the deviation from pure thirds reduces to 13.7*c*.

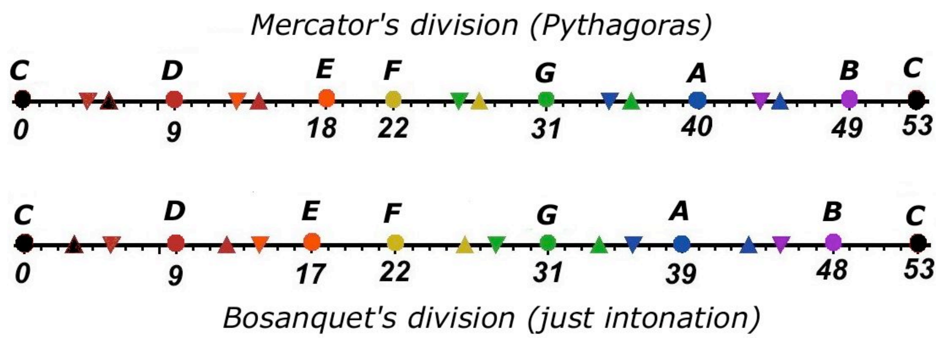

Figure 1: Comparison between Mercator and Bosanquet 53 degrees division. The first alterations are also drawn: sharps are represented with triangles pointing up; flats with triangles pointing down.

53 happens to be also a good number to get an acceptable frame for just intonation, as envisaged by R.H.M.Bosanquet [2]. Each degree in this division measures $d^{(53)} = 22.6c$ (half-way between the Pythagorean and Zarlino commas). In Fig.1 we show the comparison between the two cycles. Note, in particular, the striking difference in the location of enharmonic notes (triangles pointing up and pointing down) between the two schemes. In Mercator's division the chromatic semitone is 5 degrees, larger than the diatonic one (4 degrees). On the other hand, in the Bosanquet case, they are separated by 2 degrees (~ 45*c*), with the sharp note being lower than its flat neighbour.

Actually, the difference between enharmonic notes in the just intonation framework stems from the difference between an octave and three consecutive pure major thirds. This is known as the *diesis* (or *great diesis* or *minor diesis*) or even as the *wolf-comma*:

3 Throughout this paper we shall be concerned with *generalized equal temperaments* that consist of the division of the octave in *m* equal parts (or *degrees*). To refer to an *m*-division, abbreviations are often used such as *m*EDO or *m*-TET. In the following we shall simply refer to it as *m*-division.

$$C_W = \frac{2}{(5/4)^3} = \frac{128}{125} = 1.024 \Rightarrow I(C_W) = 41.1c \tag{7}$$

The three commas are related, since three syntonic commas minus a Pythagorean comma exactly yield the wolf-comma. We give a circular representation of this relation in Fig.2. On a circle it is natural to measure musical intervals by means of angular degrees: so, pure thirds are 115.89°, Pythagorean thirds are 122.34°. Their difference (6.45°) is the syntonic comma in these units. Starting from C, after three steps, we get B# (dashed path) or $B\#_{pyt}$ (dotted path), the angle between them amounting to three syntonic commas. On the other hand, the angle between C and $B\#_{pyt}$ (7.04°) is the Pythagorean comma. The difference is just the wolf-comma (12.32°). In fact, in Bosanquet's 53-approximation the enharmonic gap is twice a mean (Pythagoras-Zarlino) comma.

Furthermore in Mercator's temperament on Pythagorean scales, each degree gets its own precise musical name. Splitting alterations between sharps and flats symmetrically, one could go as far as up to 4 sharp and down to 4 flat. Synonymy occurs only at the far border of enharmonic modulations where sharps and flats coalesce. For instance degree=20 corresponds to *C####* as well as to *Abbbb*.

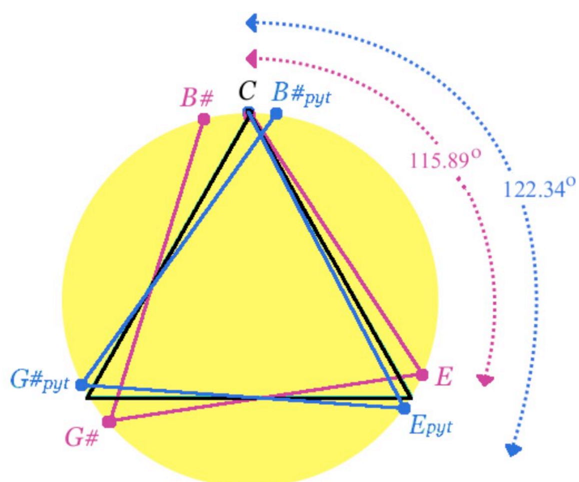

Figure 2: Circular representation of a chain of three thirds illustrating the relation among the three commas. The dashed path follows pure thirds. The dotted path follows Pythagorean thirds. The equilateral triangle represents the same path with thirds in 12-division (120° each).

On the other hand, in Bosanquet's just intonation system, it is even cumbersome to supply the 53 notes with names. As an example, a different notation (such as *E*1 or *E*, see [2]) should be used for the degree = 17 in Fig.1 instead of *E*, to be distinguished by the next *E* at the degree= 18, (roughly) a syntonic comma higher. Further complications arise from the fact that both the intervals for diatonic and chromatic semitones do not have fixed values (see Fig.1).

3. The mean-tone and the 31-cycle

Since the sixteenth century it has been sometimes thought that, in the just intonation framework, the existence of two distinct tones (the major tone (9/8) and the minor tone

(10/9), whose ratio is just the syntonic comma=81/80) prevents the establishment of a system suitable for transpositions and modulations.

The best solution may be traced back to Pietro Aron [3]: the syntonic comma is split in two equal parts so to get a *mean-tone* of frequency ratio:

$$T_M = \frac{10}{9}\left(\frac{81}{80}\right)^{1/2} = \frac{9}{8}\left(\frac{80}{81}\right)^{1/2} = \left(\frac{5}{4}\right)^{1/2} \qquad (8)$$

and the major thirds kept pure. This task is achieved by tempering the fifths by 1/4 syntonic comma:

$$F_M = \frac{3}{2}\left(\frac{81}{81}\right)^{1/4} = 1.49535 \Rightarrow I(F_M) = 696.6c \qquad (9)$$

i.e. narrowing each pure fifth ($I(3/2) = 702.0c$) by $5.4c$. This is still a tolerable amount and leaves triads quite harmonious.

Eq.(9) defines the *(quarter-comma) meantone temperament.*[4] A tour of 12 such fifths leads to a couple of enharmonic notes, a wolf-comma apart. For instance, starting from C, 8 fifths up and 4 fifths down (and due octave displacements) yields the enharmonic interval G#–Ab. With 12 notes available in an octave one has to give up one of the two, remaining with a bad interval (usually G#–Eb) known as the *wolf-fifth*. So the meantone temperament (with 12 notes) works quite well in central keys but the *wolf-fifth* heavily limits the freedom of modulation. Of course one can continue adding fifths, providing pairs of enharmonic notes, and populate the octave with more and more notes, but the process, as for pure fifths, never ends.

Now the question: how many (equal) degrees are needed in an octave to establish a sufficiently good approximation to meantone temperament? In what follows we shall answer the question in three different ways.

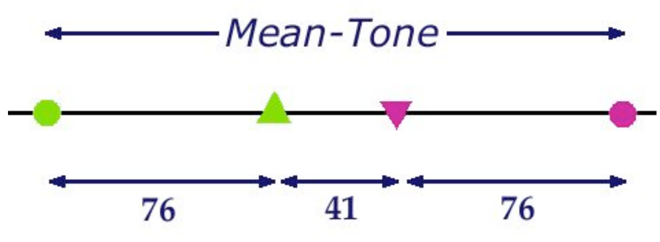

Figure 3: The splitting of a mean-tone into chromatic and diatonic semitones in meantone temperament. The mean-tone is $193c$ wide. The chromatic and diatonic semitones are $76c$ and $117c$ respectively. Their interval-ratio is $I(DS_M)=I(CS_M) = 1.54$.

Firstly, let us analyze how a mean-tone, in meantone temperament[5], is split by the diatonic and chromatic semitones (see Fig.3). The mean-tone is 193 cents wide while the chromatic and the diatonic semitones are respectively 76 and 117 cents wide[6]. The ratio between these

[4] We shall use the symbols: F_M, T_M, DS_M, CS_M for the frequency ratios of the fifth, mean-tone, diatonic semitone, chromatic semitone respectively in the meantone temperament.

[5] The specification is not superfluous, since the term *meantone* is used in general whenever an interval of major third, even not pure, is divided in two equal parts.

[6] Recall that, in meantone temperament, both the diatonic and chromatic semitones are larger than those in the just intonation (16/15 and 25/24 respectively) by 1/4 syntonic comma.

two intervals is $I(DS_M)/I(CS_M) = 1.54$. So, to answer the question, we should find two (possibly small) integers n_D and n_C such that:

$$\frac{n_D}{n_C} \approx \frac{I(DS_M)}{I(CS_M)} = 1.54 \tag{10}$$

Since there are 5 mean-tones and 2 diatonic semitones in an octave, the total number of degrees will be:

$$m = 5\,(n_D + n_C) + 2\,n_D \tag{11}$$

Indeed the m= 12 equal temperament corresponds to the choice $n_D = n_C = 1$, which is not a good approximation to Eq.(10). A bad approximation is also obtained taking $n_D = 2$, $n_C = 1$, i.e. dividing the mean-tone in 3 equal parts (m = 19). Instead one gets a fine approximation for $n_D = 3$ and $n_C = 2$ that corresponds to the division of the mean-tone in 5 equal parts, three of them giving the diatonic semitone, two the chromatic semitone and one fairly reproducing the interval between enharmonic notes[7]. So, the division of the octave in 31 equal parts (see Eq.(11)) appears as the best possibility from this point of view[8].

Secondly, we may proceed as for chains of pure fifths (see Eq.(2)) substituting them with the *meantone-tempered fifths* (see Eq.(9))[9], and look for numbers Z_m approaching 1:

$$Z_m = \left[\frac{3}{2}\left(\frac{80}{81}\right)^m\right] \Big/ 2^n \approx 1 \tag{12}$$

The first solution to Eq.(12) is (once more) $m = 12$; $n = 7$, which now gives:

$$Z_{12} = 0.9765625 = (1.024)^{-1}$$

i.e. just the (inverse of the) wolf-comma. Distributing it uniformly among the 12 (meantone-tempered) fifths leads back to the 12-degrees equal temperament, which, from this point of view, may be considered as something intermediate between the pure Pythagorean system and the meantone-tempered one. However, as an approximation to the meantone temperament, this compromise cannot be accepted, since, as already remarked in section 1, major thirds, though better than Pythagorean ones, are still too wide to be harmonious.

So we must go on and try the next solution: $m = 31$; $n = 18$:

$$Z_{31} \approx 0.996501 = (1.003511)^{-1} \;\rightarrow\; I(Z_{31}) = -6.1c \tag{13}$$

This is a small interval. Distributing it among 31 (meantone-tempered) fifths amounts to enlarging each of them by only 0.2c, so we obtain:

$$F^{(31)} = 1.49552 \;\rightarrow\; I(F(31)) = 696.8c \tag{14}$$

These fifths are narrowed by 5.2c with respect to pure fifths (instead of 5.4c; compare Eq.(9)). Since major thirds are built up in four steps, they will be only 0.8c wider than pure thirds, an imperceptible amount as emphasized by Huygens[10] in his *Letter concerning the harmonic cycle* [4].

Thus, a chain of fifths tempered as in Eq.(14) will close itself in a cycle after 31 steps and yield a 31-degree equal temperament that provides an excellent approximation to the meantone temperament.

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[7] In fact the wolf-comma ($I(C_W) = 41.1c$) is approximated even better within this division (38.7c) than in the 53-Bosanquet division (45.2c, see Sec.2).

[8] The next possibility would be the couple of prime numbers (n_D =17, n_C =11) which provides an extremely accurate solution to Eq.(9) but yields too many degrees (m = 174).

[9] This is the method produced in the essay [5].

[10] Huygens specifies this amount as about 1/28 of a (syntonic) comma.

The third way will be the subject of the next section.

4. Chains of pure thirds

Up to now we have being setting a certain number of degrees in the octave using sequences of fifths. We have looked for tempered fifths whose chains close themselves in a cycle. Since meantone temperament privileges the major third as a harmonic interval, it seems natural to face the stated question starting from chains of thirds, instead of fifths, in a totally analogous way.

First of all it is clear that, as for pure fifths, a chain of consecutive pure thirds never brings us back to the starting tone. In fact we may write an equation similar to Eq.(1):

$$\left(\frac{5}{4}\right)^m \neq 2^n \tag{15}$$

and try to find integer numbers (*m; n*) such that:

$$H_m = \frac{(5/4)^m}{2^n} \approx 1 \tag{16}$$

By the way, as chains of meantone-tempered fifths yield pure thirds, one may wonder whether using chains of pure thirds, as a method for generating tones, may lead to the same notes (possibly after a sufficiently large number of steps) as those of the meantone temperament. The answer is negative, since, for example, for any integer numbers (*m; n*):

$$\left(\frac{5}{4}\right)^m \Big/ 2^n \neq F_M \equiv \frac{3}{2}\left(\frac{80}{81}\right)^{1/4} \tag{17}$$

Eq.(17) states that the meantone-tempered fifth F_M can never be recovered by chains of pure thirds. In order to get something in the neighbourhood of F_M one has to climb at least 8 steps (i.e. m = 8 and n = 2):

$$H_8 = \left(\frac{5}{4}\right)^8 \Big/ 2^2 \approx 1.490116 \Rightarrow I(H_8) = 690.5c \tag{18}$$

that is 6.1c below F_M and 11.5c below the pure fifth.

Coming back to Eq.(16), the first obvious solution is

$$H_3 = \left(\frac{5}{4}\right)^3 \Big/ 2 \equiv Z_{12} = C_W^{\ -1} \tag{19}$$

i.e. again the (inverse of the) wolf-comma. To close the chain, each third should be made wider by $I(C_W)/3$ = 13.7c and one gets again the usual equal-tempered thirds that divide the octave in three equal parts (the equilateral triangle of Fig.2).

Next more precise solutions are:

$$H_{28} = \left(\frac{5}{4}\right)^{28} \Big/ 2^9 = 1.009742 \rightarrow I(H_{28}) = 16.8c \tag{20}$$

$$H_{31} = \left(\frac{5}{4}\right)^{31} \Big/ 2^{10} = 0.986076 = (1.01412)^{-1} \rightarrow I(H_{31}) = -24.3c \qquad (21)$$

These suggest a division of the octave in 28 or 31 equal parts respectively.

Let us examine these two possibilities. At first sight a 28-division looks like a favourable chance for an excellent solution, because of the smallness of $I(H_{28})$. In fact, to close the chain, each third should be tempered only slightly (-0.6*c*). In this case each degree is $d^{(28)} = 42.9c$ wide and the interval for a fifth amounts to 16 degrees, i.e. −16.2*c* flatter than a pure fifth, badly out of tune. There is a further drawback: the interval for major thirds is 9 degrees, an odd number that does not allow the splitting into two equal mean-tones.

So we are left with the second possibility. Now the degree is $d^{(31)} = 38.7c$, the major thirds correspond to 10 degrees and the fifths to 18 degrees. Note that the "comma" H_{31} defined in Eq.(21) corresponds to an interval quite similar in magnitude to the Pythagorean comma but of negative sign. This means that we should slightly widen the pure thirds in order to close the chain. Indeed, distributing uniformly H_{31} among the 31 thirds amounts to widening each of them of $||I(H_{31})| /31 = 0.8c$ (compare the discussion at the end of Section 3). With regard to the fifths, we recall that in Eq.(14) we saw they are 5.2*c* narrow (a little bit better than the meantone-tempered fifths – see Section 3). Note that in Section 3 we got $F^{(31)}$ by means of a further temperament of the already tempered meantone-fifths F_M. In the present scheme, on the other hand, we recover the same fifth after 8 steps in the chain of thirds. In fact H_8 (see Eq.(18)) is quite a narrow fifth when built up with pure thirds, but it gains 6.3*c* from the slight temperament of the thirds, so to coincide with $F^{(31)}$.

Conclusions

The Huygens-cycle has many valuable features. It displays a sufficient number of notes to allow a distinction between double sharps and double flats[11] so that it provides the conceptual melodic and harmonic framework of baroque, classic and romantic western music. Moreover the 26th degree happens to coincide nearly exactly with the 7th harmonic (since $25 \cdot d_{31} = 967.7c$ and $I(7/4) = 968.8c$)[12], so opening new perspectives in musical languages [5]. In fact the exploitation of the 7th harmonic was advocated in the 18th century by scientists [6] and musicians [7] and more recently by the physicist and musician Adriaan Fokker [8, 9] and others.

Though various divisions of the octave in many parts have been explored and tested since the 16th century, the division in 31 parts was refused by people like Salinas and Mersenne. Huygens attributed this misunderstanding to the inability of exactly tuning 31 equal degrees without the help of logarithms, not yet known at those times. The interest of Huygens in tuning and temperament goes back to 1661. About the same time Lemme Rossi [10] published a treatise where the 31-division is explicitly described. It appears that Huygens did not know about this work. However his main point was to find a division of the octave providing the best approximation to meantone temperament, i.e. precisely the point of view adopted in the present paper.

We do not know which method Huygens exploited to get the solution (in fact Fokker [8] suggests the first one of those outlined here); in any case we would think it reasonable to refer to H_{31}, the analogue of the Pythagorean Comma for chains of pure thirds, as the "*Huygens comma*".

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[11] With the exclusion of *Cbb; Fbb*, which coincide with *A##;D##* (28thand 10th degree respectively) and *E##;B##* with *Gbb;Dbb* (15th and 2nd degree respectively).

[12] For comparison, the *A#* (≡ *Bb*) in 12-degree equal temperament amounts to 1000*c*.

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